$p \wedge q$

$p \leftrightarrow q$

DBM20153-DISCRETE MATHEMATICS

Chapter 3

$\neg p$

$p \rightarrow q$

$\equiv$

$p \vee q$

Chapter 3: Basic Logic

3.1 Explain Propositional Logic

- 3.1.1 Explain compound proposition
  - a) Negation
  - b) Conjuction
  - c) Disjunction
  - d) Conditional
    - i) Converse
    - ii) Contrapositive
    - iii) Inverse
  - e) Biconditional

Chapter 3: Basic Logic

3.1.2 Construct truth table

a) Tautology

b) Contradiction

c) Contingency

3.1.3 Write proposition logic in English

3.2 Explain predicate logic

3.2.1 Identify compound statement in predicate logic

3.2.2 Explain quantifier

a) Universal Quantifier and Existential Quantifier

b) Quantified statements

3.1 Explain propositional logic

A **proposition** (or **statement**) is a sentence that is either **True** or **False**.

- **Letters** are used to denote **propositional variables.**

  Example: *p, q, r, s*, …
- The **truth value**of a proposition
  - True proposition (T)
  - False proposition (F)

EXAMPLES:

Proposition:

- $5 + 3 = 8$ **True**
- $10 / 2 = 4$ **False**
- 5 is an even number
- Today is Wednesday

Non-proposition:

- Where do you live **Question**
- Please answer the question correctly **Instruction**
- $x < 10$

Unknown value of *x*

Exercise A

1. Which of these sentences are proportions? What are the truth values of those that are propositions?

   a) Kuala Lumpur is the capital of Malaysia.
   b) $8 + 2 = 10$
   c) $-48 < -47$
   d) Do you want to go to a cinema?
   e) Answer this question.
   f) $x + 2 = 18$
   g) Today is Monday.
   h) Move this table to the other room

3.1.1 EXPLAIN COMPOUND PROPOSITION

Connectives in Proposition Logic

Conjunction	Conditional	Disjunction	Biconditional	Negation
$p \land q$	$p \to q$	$p \lor q$	$p \leftrightarrow q$	$\neg p$

CONJUNCTION $p \wedge q$ (READ AS p AND q)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:
p:5 + 3 = 8
q:A decade is 10 years.

$p \wedge q$
5 + 3 = 8 and a decade is 10 years.

Another answer:
$p \wedge q$
5 + 3 = 8 but a decade is 10 years.

EXERCISE B

1. Determine whether the statements are true (T) or false (F).

a) $3 + 2 = 5$ and $4 + 4 = 8$

b) Changlun is in Perlis and Alor Setar is in Kedah.

c) $-48 < -47$ and $25 + 3 = 38$

d) Duck has 4 legs and cat has wings.

e) $4x + 3x = 5x$ and $\frac{5}{4} + \frac{3}{7} = \frac{47}{28}$

DISJUNCTION $p \vee q$ (READ AS p OR q)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:
p:5 + 3 = 8
q:A decade is 10 years.

$p \vee q$
5 + 3 = 8 or a decade is 10 years.

Exercise C

1. Determine whether the statements are true (T) or false (F).

a) $3 + 2 = 5$ or $4 + 4 = 8$
b) Changlun is in Perlis or Alor Setar is in Kedah.
c) $-48 < -47$ or $25 + 3 = 38$
d) Duck has 4 legs or cat has wings.
e) $4x + 3x = 5x$ or $\frac{5}{4} + \frac{3}{7} = \frac{47}{28}$

NEGATION ¬P (READ AS NOT P)

p	$\neg p$
T	F
F	T

Example:
Write the negation for each of the following propositions.
(a) $5 + 3 = 8$
(b) A decade is 10 years.

Answer:
(a) $5 + 3 \neq 8$
(b) A decade is not 10 years.

Exercise D

1.What is the negation of each of these propositions?

a) Today is Tuesday.
b) China is in Asia
c) 2 + 1 = 3
d) All kittens are cute.
e) No prime number is even.
f) Some cookies are sweet.
g) Every lawyer uses logic.
h) No bullfrog has lovely eyes.

CONDITIONAL $p \rightarrow q$ (READ AS IF p, THEN q)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:
p: I do my homework.
q: I get my allowance.

$p \rightarrow q$
If I do my homework, then I get an allowance.

Another ways to express conditional statement:

- If p, q
- p is sufficient for q
- q if p
- q when p
- a necessary condition for q is p

CONVERSE, CONRAPOSITIVE AND INVERSE

- Starting with a conditional statement statement $\boldsymbol{p \rightarrow q}$, we can form some new conditional statement.

- There are three related conditional statements that occur so often which is CONVERSE, CONRAPOSITIVE AND INVERSE

The proposition $(q \rightarrow p)$ is called the **CONVERSE of** $(p \rightarrow q)$

The proposition $(\neg q \rightarrow \neg p)$ is called the **CONRAPOSITIVE of** $(p \rightarrow q)$

The proposition $(\neg p \rightarrow \neg q)$ is called the **INVERSE of** $(p \rightarrow q)$

EXAMPLE 9

What are the contrapositive, the converse, and the inverse of the conditional statement
“The home team wins whenever it is raining.”?

Solution:
The original statement can be rewritten as
“If it is raining , then the home team wins”

- *Contrapositive: “If the home team does not win, then it is not raining”*
- *Converse: “If the home team wins, then it is raining”*
- *Inverse: If it is not raining, then the home team does not win.”*

Exercise E

1. Let p be “ It is cold” and q be “ It is raining”. Give a simple sentence which describes each of the following statements:
   a) $p \rightarrow q$
   b) $q \rightarrow \neg p$
   c) $\neg q \rightarrow \neg p$

2. State the converse, contrapositive, and inverse of the following conditional statements
   “I come to class whenever there is going to be a quiz”.

Biconditional $p \leftrightarrow q$ (read as p if and only if q)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example:
p: You passed the Discrete Mathematics exam.
q: You scored 40% or higher.

$p \leftrightarrow q$
You passed the Discrete mathematics exam if and only if you scored 40% or higher.

Another ways to express biconditional statement:

- p is necessary and sufficient for q
- if p then q, and conversely
- p iff q

Exercise F

1. Let p be " It is cold" and q be " It is raining". Give a simple sentence which describes each of the following statements:

   a) $p \leftrightarrow q$

   b) $q \leftrightarrow \neg p$

Exercise G (Question 1)

Which of these sentences are propositions? State the truth value of those that are propositions.

a) If it snows, then the schools are closed.
b) x + 2 is positive.
c) Take the umbrella with you.
d) No prime number is even.
e) A triangle is not a polygon. (**polygon is a closed path*)

Exercise G (Question 2)

Let p and q be the propositions

p: Andy is going to Brunei

q: Andy is having a holiday.

Express each of these propositions as an English sentence.

a) $\neg p$

b) $q \vee \neg p$

c) $\neg p \wedge \neg q$

d) $p \leftrightarrow q$

Exercise G (Question 3)

Represent the sentences below as propositional expressions:

a) Tom is a math major but not computer science major.
b) You can either stay at the hotel and watch TV or you can go to the museum
c) If it is below freezing, it is also snowing.

Exercise G (Question 4)

Determine whether each of these statements is true or false.

a) If $1 + 1 = 2$, then $2 + 2 = 5$
b) If monkeys can fly, then $1 + 1 = 3$
c) $2 + 2 = 4$ if and only if $1 + 1 = 2$
d) $0 > 1$ if and only if $2 > 1$

PAST YEAR QUESTIONS SESI II: 2021/2022

QUESTION 1

Identify each of the following sentence whether a proposition or not and state the truth value.

a) Is 2 a positive number? (1 mark)

b) A triangle is a three-sided polygon (2 marks)

c) $4 + 9 > 9$ (2 marks)

PAST YEAR QUESTIONS SESI II: 2021/2022

<u>QUESTION 2</u>

Let p denote “ John is rich” and q denote “ John is happy”. Write each statement in symbolic form using p and q.

a) If John is rich, then he is unhappy.
b) John is neither rich nor happy.
c) It is necessary to be poor in order to be happy.
d) John is unhappy if and only is he is poor.
e) To be poor is to be unhappy.

(10 marks)

3.1.2 Method for constructing truth tables

A logical statement/proposition having n component statements will have 2^n rows in its truth table.

<u>Example:</u>

If we have two propositions (p and q), therefore, we will have $2^2 = 4$ rows.

If we have three propositions (p, q and r), therefore we will have $2^3 = 8$ rows.

Examples:

Two propositions

p	q
T	T
T	F
F	T
F	F

Three propositions

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Example

Construct the truth table for

$$p \wedge (\neg p \vee \neg q)$$

Solution

First, we know that there are 2 propositions involve

Therefore, there will be 4 rows in the truth table.

Next, there are brackets and 2 negations
(2 additional columns)
connected with the symbol ∨ (1 additional column)
and the last column is what we need to construct.

Construct the truth table for

$$p \wedge (\neg p \vee \neg q)$$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge (\neg p \vee \neg q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

Exercise H

Construct the truth table for each of the following:

a) $\neg p \wedge q$

b) $(\neg p \vee q) \rightarrow \neg q$

c) $p \wedge (\neg q \vee r)$

d) $\neg p \leftrightarrow q \vee r$

TAUTOLOGY, CONTRADICTION & CONTIGENCY

TAUTOLOGY

A proposition P (*p, q, r,* ……) is a **tautology** if it contains **only T in the last column** of its truth table.
In other word they are true for any truth values of their variables.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

The proposition p V ¬p is a tautology.

CONTRADICTION

A proposition P (*p, q, r,* ……) is a **contradiction** if it contains **only F in the last column** of its truth table. In other words they are false for any truth values of their variables.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

The proposition p ∧¬p is a contradiction.

CONTINGENCY

A proposition P (*p*, *q*, *r*, ……) is a **contingency** if it contains both **T** and **F** in the last column of its truth table.

Example:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The proposition $p \vee q$ is a contigency.

Exercise J

1. Use the truth table to determine whether the statement is a tautology, contradiction or contingency: $((p \to q) \land p) \to q$

2. Use a truth table to show that the proposition is always true: $p \lor (q \lor \neg p)$

3. Determine whether the proposition is tautology or not: $(p \to q) \land (q \to p) \leftrightarrow (p \to \neg q)$

Chapter 3: Basic Logic

3.2 Explain Predicate Logic

- 3.2.1 Identify the compound statement in predicate logic
- 3.2.2 Explain quantifier:
  - (a) Universal and Existential
  - (b) Quantified statements

3.2 EXPLAIN PREDICATE LOGIC

- A **predicate** is a **statement that contains variables (predicate variables)** and that **may be true or false depending on the values of these variables**.
- ***P(x)*** is a predicate.

EXAMPLE

Consider the statement involving variables such as "$x > 3$".
The statement "***x* is greater than 3**" has two parts:

- The first part, the variable x is the subject of the statement.
- The second part, "**is greater than 3**" is the predicate of the statement.

We can denote the statement "*x* is greater than 3" by ***P(x)***, where ***P*** denotes the predicate "is greater than 3" and ***x*** is the variable.

If ***P(x)*** is a predicate and ***x*** has domain **D**, the truth set of ***P(x)*** is the set of all elements of **D** that make ***P(x)*** true when they are substituted for x. The truth set of ***P(x)*** is denoted $\{\, \{x \in D\} \mid P(x) \,\}$ which is read "***the set of all x in D such that P(x)***".

EXAMPLE

Let $P(x)$ denote the statement "$x > 3$". What are the truth values of $P(4)$ and $P(2)$?

<u>Solution:</u>

$P(x) = x > 3$

Substitute the values given to the predicate;

$$P(\mathbf{4}) = \mathbf{4} > 3 \quad \text{(TRUE)}$$

$$P(\mathbf{2}) = \mathbf{2} > 3 \quad \text{(FALSE)}$$

EXAMPLE

Example:

Assume a **predicate P(x)** that represents the statement x **is a prime number**.

What are the truth values of P(**2**), P(**3**), P(**4**), P(**5**), P(**6**) and P(**7**)?

<u>Solution:</u>

P(**2**) **(TRUE) since 2 is a prime number**

P(**3**) **(TRUE) since 3 is a prime number**

P(**4**) **(FALSE) since 4 is not a prime number [4 can be divided by 2)**

P(**5**) **(TRUE) since 5 is a prime number**

P(**6**) **(FALSE) since 6 is not a prime number [6 can be divided by 2 and 3)**

P(**7**) **(TRUE) since 5 is a prime number**

Exercise K

1. Let P(x) be the statement “the word x contains the letter a”. What are these truth values?

   a) P(orange)
   b) P(lemon)
   c) P(false)

2. Let P(x) be the statement “x is the states in Malaysia that starts with the letter P”. Find the truth set of P(x), where the domain is all the state in Malaysia.

Exercise K (cont…)

3.Let P(x) be the statement $x = x^2$. If the domain consists of the integers, what are the truth values?

a)P (0)

b)P (1)

c)P (2)

d)P (-1)

3.2.1 Identify the compound statement in predicate logic.

A predicates can have **more arguments** which represent the **relations between objects.**

<u>Example:</u>

- Older(John, Peter) denotes ‘John is older than Peter’

 – this is a **proposition** because it is **either true or false**

- Older(x,y) - ‘x is older than y’

 – **not a proposition**, but after the substitution it becomes one

EXAMPLE

Let Q(x,y) denote ‘x + 5 > y’. Answer all questions below.

Solution:

– Is Q(x,y) a proposition? **No** (cannot detemine whether it is true or false)

– Is Q(3,7) a proposition? **Yes** (It is true)

– Is Q(3,y) a proposition? **No** (cannot detemine whether it is true or false since y is unknown)

– What is the truth value of:

- Q(3,7) **T** (True since 3+5 > 7; 8 > 7)
- Q(1,6) **F** (False since 1+5 > 6; 6 > 6)
- Q(2,2) **T** (True since 2+5 > 2; 7 > 2)

IMPORTANT

- Statement P(x) is **not a proposition** since there are more objects it can be applied to

- **This is the same as in propositional logic… But the difference is:**
  - Predicate logic allows us to explicitly manipulate and substitute for the objects
  - **Predicate logic permits** quantified sentences where **variables are substituted** for statements about the group of objects

Exercise L

1. Let Q(x,y) denote the statement "x=y+3". What are the truth values of the propositions Q(1,2) and Q(3,0)?

2. Let A(c,n) denote the statement “Computer c is connected to network n,” where c is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the truth values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

3. Let Q(x, y, z) denote the statement “$x^2 + y^2 = z^2$”. What is the truth value of Q(3, 4, 5)? What is the truth value of Q(2, 2, 3)?

3.2.2 COMPARE THE TYPE OF QUANTIFIER (UNIVERSAL & EXISTENTIAL)

Universal quantifier (∀x)

- The universal quantification of P(x): "***P(x) is true for all values of x in the domain of discourse.***"
- The notation **∀ x P(x)** denotes the universal quantification of P(x), and is expressed as **for every *x*, P(x)**.
- **¬ (∀ x P(**x**)**) write as "**It is not true that all P(x)**"
- **∀ x ¬ P(x)** write as "**All x are not P(x)**"

Existential quantifier (∃x)

- The **existential quantification** of P(x): "***There exists an element in the domain (universe) of discourse such that P(x) is true.***"
- The notation ∃ **x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.
- ¬ (∃ x P(x)) write as "**It is not true that some P(x)**"
- ∃ **x ¬ P(x)** write as "**Some x are not P(x)**"

ADDITIONAL INFORMATION

The truth value:

$(\exists\ x\ P(x)) \equiv \forall\ x\ \neg\ P(x)$

$\neg\ (\forall\ \mathbf{x}\ \ \mathbf{P}(\mathbf{x})) \equiv \exists\ \mathbf{x}\ \neg\ \mathbf{P}(\mathbf{x})$

3.2.2 IDENTIFY THE QUANTIFIED STATEMENTS

<u>Example:</u>

Let D={1, 2, 3, 4, 5} and consider the statement ∀x∈D, $x^2 \geq x$.

Show that this statement is true.

<u>Solution:</u>

Check that "$x^2 \geq x$" is true for each individual x in D.

$\mathbf{1^2 \geq 1}$, $\mathbf{2^2 \geq 2}$, $\mathbf{3^2 \geq 3}$, $\mathbf{4^2 \geq 4}$, $\mathbf{5^2 \geq 5}$

$\mathbf{1 \geq 1}$, $\mathbf{4 \geq 2}$, $\mathbf{9 \geq 3}$, $\mathbf{16 \geq 4}$, $\mathbf{25 \geq 5}$

Hence "$\forall\, \boldsymbol{x} \in \boldsymbol{D}, x^2 \geq x$ " **is true**.

EXAMPLE

Let P(x) be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x\, P(x)$ where the domain consists of all real numbers?

Solution:

Because P(x) is true for all real numbers x, the quantification **$\forall x\, P(x)$ is true.**

EXAMPLE

Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall\, x\, Q(x)$ where the domain consists of all real numbers?

<u>Solution:</u>

$Q(x)$ is not true for every real number x. Example: $Q(3) = 3 < 2$ is FALSE.

That is x = 3 is a counterexample for the statement $\forall\, x\, Q(x)$. Thus $\forall\, x\, Q(x)$ is false.

EXAMPLE

Consider the statement $\forall\, x \in R,\, x^2 \geq x$. Find a counterexample to show that this statement is false.

<u>Solution:</u>

Let x= $\frac{1}{2}$. Then x is in R and $(\frac{1}{2})^2 = \frac{1}{4} \ngeq \frac{1}{2}$. Hence "$\forall\, x \in R,\, x^2 \geq x$ "is FALSE.

That is x = $\frac{1}{2}$ is a counterexample for the statement ∀ x ∈ R, $x^2 \geq x$.

Thus, ∀ x ∈ R, $x^2 \geq x$ is false.

EXAMPLE

Consider the statement $\exists m \in Z$ such that $m^2=m$. Show that this statement is true.

<u>Solution:</u>

Observe that $1^2 = 1$. Thus "$m^2 = m$" is **true for at least one integer m**.

Hence "$\exists m \in Z$ such that $m^2 = m$" is TRUE.

WRITE A WELL-FORMED PREDICATE LOGIC IN ENGLISH

Example 1:

Let P(x) denotes the statement “x is taking a mathematics course”. The domain of discourse is the set of all students. Write each propositions in words.

a) $\forall x\, P(x)$
b) $\exists x\, P(x)$
c) $\neg(\exists x\, P(x))$
d) $\forall x\, \neg P(x)$

SOLUTION

- $\forall x\, P(x)$

  All students are taking a Mathematics course.

- $\exists x\, P(x)$

  Some students are taking a Mathematics course.

- $\neg(\exists x\, P(x))$

  It is not true that some students are taking a Mathematics course.

- $\forall x\, \neg P(x)$

  All students are not taking a Mathematics course.

EXAMPLE 2

Let the universe be the set of airplanes and let F(x,y) denote “x flies faster than y”. Write each propositions in words.

a) $\forall x \forall y F(x,y)$

“Every airplane is faster than every airplane”

b) $\forall x \exists y F(x,y)$

“Every airplane is faster than some airplane”

c) $\exists x \forall y F(x,y)$

"Some airplane is faster than every airplane"

d) $\exists x \exists y F(x,y)$

"Some airplane is faster than some airplane”

TRANSFER THE TRANSLATION WITH QUANTIFIERS

Example:

Let $P(x)$ be the statement "x knows kung fu" and $Q(x)$ be the statement "x knows karate" where the domain consists of all adults in your neighborhood. Write the following sentences using predicates, quantifiers and logical connectives.

- There is an adult in your neighborhood who knows kung fu and karate.
- There is an adult in your neighborhood who knows kung fu but not karate.
- Every adult in your neighbourhood knows kung fu or karate.
- No adult in your neighbourhood knows kung fu or karate.

SOLUTION

- There is an adult in your neighbourhood who knows kung fu and karate.

$$\exists x(P(x) \land Q(x))$$

- There is an adult in your neighbourhood who knows kung fu but not karate.

$$\exists x(P(x) \land \neg Q(x))$$

- Every adult in your neighbourhood knows kung fu or karate.

$$\forall x\ (P(x) \lor Q(x))$$

- No adult in your neighbourhood knows kung fu or karate.

$$\forall x\ \neg(P(x) \lor Q(x)) \equiv \neg\exists\ x\ (P(x) \lor \neg Q(x))$$

ADDITIONAL EXAMPLE

Problem:

Express the statement "Not everybody can ride a bike" as a logical expression.

Solution:

- Let $P(x)$ = "x can ride a bike."
- The statement "everybody can ride a bike," can be expressed as $\forall \boldsymbol{x}\, \mathbf{P}(\boldsymbol{x})$.
- We want the negation of this, which is $\neg\forall \boldsymbol{x}\, \mathbf{P}(\boldsymbol{x})$.
- Another way to say this is "There is somebody that cannot ride a bike," which can be expressed as $\exists \boldsymbol{x}\, \neg\mathbf{P}(\boldsymbol{x})$.

ADDITIONAL EXAMPLE

Problem:

Express the statement “Nobody can fly.” as a logical expression.

Solution:

- Let P(x) = “x can fly.”
- The statement “somebody can fly,” can be expressed as $\exists$ **x P(x)**.
- We want the negation of this, which is **¬∃ x P(x)**.
- Another way to say this is “Everybody can not fly,” which can be expressed as $\forall$ **x ¬P(x)**.

Exercise M

Translate the specifications into English sentences where P(x) be the predicate "x must take a discrete mathematics course" and let Q(x) be the predicate "x is a computer science student". The universe of discourse for both P(x) and Q(x) is all students.

(a) $\forall x\ (Q(x) \land P(x))$
(b) $\exists x\ (P(x) \to Q(x))$
(c) $\neg(\forall x\ Q(x) \to P(x))$

Exercise N

1. Let S(x,y) be the predicate “x is expensive than y” and let the universe of discourse be the set of cars. Express the following in sentences:
   (a) $\exists x \exists y\, S(x,y)$
   (b) $\exists x\, \neg S(x,\text{Mercedes})$
   (c) $\neg\forall x\, \exists y\, S(x,y)$

Let P(x): 'x likes sport'.

Let Q(x): 'x can speak English'.

2. The domain for x is the set of all lecturers in Polytechnic.Translate symbolically the following expressions:
   (a) Some lecturers in Polytechnic like sport and can speak English.
   (b) Every lecturers in Polytechnic like sport if they cannot speak English.